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Noncommutative Conformal Field Theory in the Twist-deformed context

Fedele Lizzi[2] **and Patrizia Vitale**[2]

[2] *Dipartimento di Scienze Fisiche, Università di Napoli Federico II*
and *INFN, Sezione di Napoli*
Monte S. Angelo, Via Cintia, 80126 Napoli, Italy
`fedele.lizzi@na.infn.it`, `patrizia.vitale@na.infn.it`

Abstract

We discuss conformal symmetry on the two dimensional noncommutative plane equipped with Moyal product in the twist deformed context. We show that the consistent use of the twist procedure leads to results which are free from ambiguities. This lends support to the importance of the use of twist symmetries in noncommutative geometry.

1 Introduction

The presence of a noncommutative structure of space (or spacetime), can be described by a nontrivial commutation relation among the coordinates, or more generally by the presence of a deformation of the product among functions, the deformed $\star$ product. The presence of the noncommutativity parameter also means that spacetime symmetries are in general broken, at least as classical symmetries. It is in fact possible to consider a *quantum symmetry* for these theories. This means that the Lie Algebra is deformed into a noncocommutative Hopf algebra, and in this case it is still possible to have a symmetry, albeit of a new, quantum, type.

While most of the interest has been for the Lorentz or Poincaré symmetries, in this paper we will concentrate our attention on the two-dimensional conformal symmetry [1].

In this paper we will consider a field theory on the noncommutative plane $\mathbb{R}^2_\theta$. This is the plane equipped with a noncommutative star product for which

$$x_1 \star x_2 - x_2 \star x_1 = i\theta \tag{1.1}$$

with $\star$ the Grönewold-Moyal star product

$$(f \star g)(x) = e^{\frac{i}{2}\theta(\partial_{x_0}\partial_{y_1} - \partial_{x_1}\partial_{y_0})} f(x) \cdot g(y)|_{x=y} \tag{1.2}$$

In [2] we showed that, by deforming the commutation relations between creation and annihilation operators appropriately, quantum conformal invariance of a simple two-dimensional field theory is preserved on the noncommutative plane. In that paper we followed the following philosophy: We assumed there would be a conformal symmetry, *and then* looked for commutation relations which could fulfill them. We found that it was necessary a deformation of the commutator. Deformations of this kind had appeared earlier in the literature (for example in [3, 4, 5], and have been not free from controversies. In [6] together with Paolo Aschieri we have given a consistent prescription for the application in the twist, which we will discuss below. In [7, 8], together with S. Galluccio we have shown as the consistent use of this procedure enables the understanding, at a physical level, of the equivalence among the Grönewold-Moyal and the Wick-Voros product. In this note we reconsider the issue of twisted conformal symmetry and show that the consistent use of the procedure developed in [6] leads to results which are free from ambiguities and consistent with [2].

2 The model

We first present, mainly to fix notation, the model in the usual commutative setting. Let us consider the two dimensional Minkowski plane in light cone coordinates. We will use

the convention in which the spacetime metric $\eta_{\mu\nu} = \mathrm{diag}(1,-1)$. Light cone coordinates are defined as $x^{\pm} = x^0 \pm x^1$. Using $\eta_{AB}dx^A dx^B = \eta_{\mu\nu}dx^\mu dx^\nu$, and $\eta^{AB} = (\eta_{AB})^{-1}$ (here A, B label lightcone indices + and -) we have

$$\begin{aligned} \eta_{++} = \eta_{--} \quad &= 0 = \quad \eta^{++} = \eta^{--}, \\ \eta_{+-} = \eta_{-+} \quad &= \tfrac{1}{2} = \quad (\eta^{+-})^{-1} = (\eta^{-+})^{-1}. \end{aligned} \tag{2.1}$$

From $x_A = \eta_{AB}x^B$ we also have the rule $x_{\pm} = x^{\mp}/2$.

We consider the simplest two dimensional conformally invariant theory, a scalar massless field theory described by the action

$$S = \int d^2x \ \partial_+\varphi\partial_-\varphi \ . \tag{2.2}$$

The classical solutions are fields split in "left" and 'right" movers $\varphi = \varphi_+(x^+) + \varphi_-(x^-)$. In the quantum theory the fields are operators with mode expansion

$$\phi(x^0, x^1) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi k_0} \left(a(k)e^{-ik^\mu x_\mu} + a^\dagger(k)e^{ik^\mu x_\mu} \right). \tag{2.3}$$

Using $k^0 = k_0 = |k^1|$ this can be rewritten as

$$\begin{aligned} \phi(x^+, x^-) \quad = \quad & \int_{-\infty}^{0} \frac{dk^1}{4\pi|k^1|} \left(a(k)e^{-i|k^1|x^+} + a^\dagger(k)e^{i|k^1|x^+} \right) \\ & + \int_{0}^{\infty} \frac{dk^1}{4\pi|k^1|} \left(a(k)e^{-ik^1x^-} + a^\dagger(k)e^{ik^1x^-} \right). \end{aligned} \tag{2.4}$$

This in turn may be rewritten as

$$\phi(x^+, x^-) = \int_{-\infty}^{\infty} \frac{dk^1}{4\pi|k^1|} \left(a_-(k)e^{-ik^1x^+} + a_+(k)e^{-ik^1x^-} \right) = \phi_+(x^+) + \phi_-(x^-) \tag{2.5}$$

where we have introduced the symbol $a_\sigma(k),\ \sigma = +,-,\ k \in (-\infty, \infty)$ related to the two sets of oscillators appearing in (2.4) (left and right movers) by $a_\sigma(k) = a(\sigma k),\ a_\sigma(-k) = a^\dagger(-\sigma k),\ k \in (0, \infty)$. The commutation relations for the creation and annihilation operators are

$$[a_\sigma(p), a_{\sigma'}(q)] = 2p\delta(p+q)\delta_{\sigma\sigma'}. \tag{2.6}$$

Then the quantum currents

$$\begin{aligned} J^+(x) \quad &= \quad 2J_-(x) = 2\partial^-\phi = i\int_0^{\infty} \frac{dk^1}{2\pi} \left(a^\dagger(k)e^{ik^1x^-} - a(k)e^{-ik^1x^-} \right) \\ J^-(x) \quad &= \quad 2J_+(x) = 2\partial^+\phi = i\int_{-\infty}^{0} \frac{dk^1}{2\pi} \left(a^\dagger(k)e^{-ik^1x^+} - a(k)e^{ik^1x^+} \right) \end{aligned} \tag{2.7}$$

generate two commuting U(1) Kaç-Moody algebras with central extension

$$\begin{aligned}\left[J^{\pm}(x), J^{\pm}(y)\right] &= -\frac{i}{\pi}\partial_{\mp}\delta(x^{\mp}-y^{\mp}),\\ \left[J^{+}(x), J^{-}(y)\right] &= 0.\end{aligned} \tag{2.8}$$

Quantum conformal invariance is proved showing that the components of the quantum stress-energy tensor

$$\begin{aligned}\Theta_{\pm\pm} &= \frac{1}{4}(\Theta_{00}\pm 2\Theta_{01}+\Theta_{11}),\\ \Theta_{+-} &= \frac{1}{4}(\Theta_{00}-\Theta_{11}) = \Theta_{-+}\end{aligned} \tag{2.9}$$

generate the conformal algebra. Tracelessness and conservation ($\partial^{\mu}\Theta_{\mu\nu}=0$) imply $\Theta_{\pm\mp}=0$ and $\partial^{\pm}\Theta_{\pm\pm}=0$. Hence $\Theta_{\pm\pm}(=\frac{\Theta^{\mp\mp}}{4})$ is a function of $x^{\mp}$ only, as in the standard case. Classically, $\Theta^{++}(x) = J^{+}(x)J^{+}(x)$. The quantum stress-energy tensor is the *normal-ordered* product

$$\Theta^{\pm\pm}(x) = \gamma : J^{\pm}(x)J^{\pm}(x) : , \tag{2.10}$$

where γ is a real number which gets fixed in the quantum theory, and normal ordering is defined as

$$\begin{aligned}: a_{\sigma}(p)a_{\sigma}(q) : &= a_{\sigma}(p)a_{\sigma}(q) \text{ if } p<q\\ : a_{\sigma}(p)a_{\sigma}(q) : &= a_{\sigma}(q)a_{\sigma}(p) \text{ if } p\geq q.\end{aligned} \tag{2.11}$$

Creation and annihilation operators obey the commutation rules

$$a_{\sigma}(p)a_{\sigma'}(q) = 2p\delta(p+q)\delta_{\sigma\sigma'}, \tag{2.12}$$

Therefore the existence of Kaç-Moody quantum current algebras is a sufficient condition to ensure conformal invariance at the quantum level. It results

$$\begin{aligned}[\Theta^{\pm\pm}(x), \Theta^{\pm\pm}(y)] &= \pm\frac{4i}{\pi}\Theta^{\pm\pm}(x)\partial_{\mp}\delta(x-y) - \frac{i}{6\pi^{3}}\partial_{\mp}^{'''}\delta(x-y)\\ [\Theta^{++}(x), \Theta^{--}(y)] &= 0\end{aligned} \tag{2.13}$$

.

3 The NC case: a review

We consider now Moyal type noncommutativity, which implies, for the fields of the theory

$$\phi(x)\star\psi(y) = \left[m_0\circ\mathcal{F}^{-1}(\phi\otimes\psi)\right](x,y) = e^{\frac{i}{2}\theta^{\mu\nu}\partial_{x_{\mu}}\partial_{y_{\nu}}}\phi(x)\psi(y). \tag{3.1}$$

Here m_0 is the ordinary pointwise multiplication while the twist operator and its inverse are

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} \quad , \quad \mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} \; ; \tag{3.2}$$

with $\frac{\partial}{\partial x^\mu}$ and $\frac{\partial}{\partial x^\nu}$ globally defined vectorfields on $\mathbb{R}^d$ (infinitesimal translations). Given the Lie algebra Ξ of vectorfields with the usual Lie bracket

$$[u,v] := (u^\mu\partial_\mu v^\nu)\partial_\nu - (v^\nu\partial_\nu u^\mu)\partial_\mu \; , \tag{3.3}$$

and its universal enveloping algebra $U\Xi$, the twist $\mathcal{F}$ is an element of $U\Xi \otimes U\Xi$. The elements of $U\Xi$ are sums of products of vectorfields, with the identification $uv - vu = [u,v]$.

We shall frequently write (sum over α understood)

$$\mathcal{F} = \mathsf{f}^\alpha \otimes \mathsf{f}_\alpha \quad , \quad \mathcal{F}^{-1} = \bar{\mathsf{f}}^\alpha \otimes \bar{\mathsf{f}}_\alpha \; , \tag{3.4}$$

so that

$$f \star g := \bar{\mathsf{f}}^\alpha(f)\bar{\mathsf{f}}_\alpha(g) \; . \tag{3.5}$$

Explicitly we have

$$\mathcal{F}^{-1} = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x^\mu}\otimes\frac{\partial}{\partial x^\nu}} = \sum \frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\ldots\theta^{\mu_n\nu_n}\partial_{\mu_1}\ldots\partial_{\mu_n}\otimes\partial_{\nu_1}\ldots\partial_{\nu_n} = \bar{\mathsf{f}}^\alpha \otimes \bar{\mathsf{f}}_\alpha \; , \tag{3.6}$$

so that α is a multi-index.

The strategy we followed in [2] was to *replace commutators with deformed commutators*

$$[a,b] \longrightarrow a \star b - b \star a \tag{3.7}$$

and impose that the KM algebra (2.8) be the same. In the commutative case this was a sufficient condition for quantum conformal invariance, thanks to Sugawara construction. Imposing that

$$\begin{aligned} J^\pm(x)\star J^\pm(y) - J^\pm(y)\star J^\pm(x) &= -\frac{i}{\pi}\partial_\mp\delta(x^\mp - y^\mp), \\ J^+(x)\star J^-(y) - J^-(y)\star J^+(x) &= 0. \end{aligned} \tag{3.8}$$

implies in turn a new commutation rule for the creation and annihilation operators

$$a_\sigma(p)a_{\sigma'}(q) = \mathcal{F}^{-1}(p,q)a_{\sigma'}(q)a_\sigma(p) + 2p\delta(p+q)\delta_{\sigma\sigma'}, \tag{3.9}$$

where

$$\mathcal{F}^{-1}(p,q) = e^{-\frac{i}{2}p\wedge q} = e^{-i\theta(|p|q-|q|p)} \tag{3.10}$$

Then we could verify, a posteriori, that the quantum conformal commutation relations (2.13) are still valid in the NC case that is

$$\begin{aligned} \Theta^{\pm\pm}(x)\star\Theta^{\pm\pm}(y) - \Theta^{\pm\pm}(y)\star\Theta^{\pm\pm}(x) &= \pm\frac{4i}{\pi}\Theta^{\pm\pm}(x)\partial_\mp\delta(x-y) - \frac{i}{6\pi^3}\partial_\mp^{'''}\delta(x-y) \\ \Theta^{++}(x)\star\Theta^{--}(y) - \Theta^{++}(y)\star\Theta^{--}(x) &= 0. \end{aligned} \tag{3.11}$$

4 The NC case: a new approach

In [2] we pointed out that the choice of the commutator of the NC theory in the form (3.7) was not canonical, but was such that the (twisted) conformal symmetry was recovered. Indeed other choices are possible, motivated by different principles, see for example [5, 10, 9].

In [6], together with Paolo Aschieri, we have established a consistent procedure to deform all bilinear maps appearing in a NC theory, among them, commutators. The guiding principle was to deform bilinear maps

$$\mu : X \times Y \to Z \tag{4.1}$$

(where X, Y, Z are vectorspaces, with an action of the twist $\mathcal{F}$ on X and Y), by combining them with the action of the twist. In this way we obtain a deformed version $\mu_\star$ of the initial bilinear map μ:

$$\mu_\star := \mu \circ \mathcal{F}^{-1} , \tag{4.2}$$

$$\begin{array}{rcl} \mu_\star : X \times Y & \to & Z \\ (\mathsf{x}, \mathsf{y}) & \mapsto & \mu_\star(\mathsf{x}, \mathsf{y}) = \mu(\bar{\mathsf{f}}^\alpha(\mathsf{x}), \bar{\mathsf{f}}_\alpha(\mathsf{y})) \ . \end{array}$$

The $\star$-product on the space of functions is recovered setting $X = Y = \mathcal{A} = \mathrm{Fun}(M)$. Without entering into the description of the general theory, which we don't need here, let us concentrate on the implications of (4.2) for commutators.

Quantum observables are operator valued functionals defined on quantum fields. As previously, we choose to work with light-cone coordinates. Let us consider for a moment the quantum currents $J^\pm(x)$: their undeformed algebra is given in (2.8). As we easily verify, their commutator is not well defined anymore, because it is not skew-symmetric, it does not satisfy the Leibniz rule nor Jacobi identity. The current algebra is trivially derived by the commutation rules for the left and right component of the field ϕ (cfr. [11] although with a different normalization)

$$\begin{aligned} [\phi_\pm(x), \phi_\pm(y)] &= \frac{i}{8\pi}\mathrm{sgn}(x^\pm - y^\pm), \\ [\phi_+(x), \phi_-(y)] &= 0. \end{aligned} \tag{4.3}$$

Analogously to $J^\pm$, $\phi_\pm(x^\pm)$ may be regarded as coordinate fields in the quantum phase space of the theory, therefore we choose them as as fundamental objects from now on. Of course their commutator shows the same pathologies as the current algebra: it is not skew-symmetric, it does not satisfy the Leibniz rule nor Jacobi identity. Following our general prescription (4.2) there is a natural commutator on the algebra of quantum

observables which is compatible with the new mathematical structures. It is obtained by composing the undeformed one with the twist, appropriately realised on operators. We find it convenient to represent the twist operator (3.2) in light cone coordinates

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{-+}(\partial_{x^-}\otimes\partial_{x^+}-\partial_{x^+}\otimes\partial_{x^-})} \tag{4.4}$$

where we have used

$$\theta^{ij}\partial_i \otimes \partial_j = \theta^{-+}(\partial_{x^-} \otimes \partial_{x^+} - \partial_{x^+} \otimes \partial_{x^-}). \tag{4.5}$$

The algebra of quantum observables, $\widehat{\mathsf{A}}$, is an algebra of functionals $G[\phi_\pm]$ on operator valued fields.

We lift the twist (4.4) to $\widehat{\mathsf{A}}$ and then deform this algebra to $\widehat{\mathsf{A}}_\star$ by implementing the twist deformation principle (4.2). We denote by $\hat{\partial}_\pm$ the lift to $\widehat{\mathsf{A}}$ of $\frac{\partial}{\partial x^\pm}$; for all $G \in \widehat{\mathsf{A}}$,

$$\hat{\partial}_\pm G := -\int dx^\pm \; \partial_\pm\phi_\pm(x)\frac{\delta G}{\delta\phi_\pm(x)} \;; \tag{4.6}$$

here $\partial_\pm\phi_\pm(x)\frac{\delta G}{\delta\phi_\pm(x)}$ stands for $\int d\ell\, \partial_\pm\phi_{\pm\ell}(x)\frac{\delta G}{\delta\phi_{\pm\ell}(x)}$, where like in (2.5) we have expanded the operator $\phi_\pm$ as $\int d\ell\, \phi_{\pm\ell}(x)\mathsf{a}(\ell)$.

Consequently the twist on operator valued functionals reads

$$\begin{aligned} \hat{\mathcal{F}} &= e^{-\frac{i}{2}\theta^{-+}\left[\int dx^-\left(\partial_-\phi_-\frac{\delta}{\delta\phi_-(x)}\right)\otimes\int dy^+\left(\partial_+\phi_+\frac{\delta}{\delta\phi_+(y)}\right)-\int dx^+\left(\partial_+\phi_+\frac{\delta}{\delta\phi_+(x)}\right)\otimes\int dy^-\left(\partial_-\phi_-\frac{\delta}{\delta\phi_-(y)}\right)\right]} \\ &\equiv \bar{f}^\alpha \otimes \bar{f}_\alpha \end{aligned} \tag{4.7}$$

and $\bar{f}$ denote the lift of $\bar{\mathsf{f}}$ to operator valued functionals. This is composed of differential operators of any order expressed in terms of $\hat{\partial}_\pm$. As we said, in $\widehat{\mathsf{A}}_\star$ there is a natural notion of $\star$-commutator, according to the general prescription (4.2)

$$[\ ,\ ]_\star = [\ ,\ ] \circ \hat{\mathcal{F}}^{-1} \tag{4.8}$$

that is

$$[F, G]_\star = [\bar{f}^\alpha(F), \bar{f}_\alpha(G)]\ . \tag{4.9}$$

This $\star$-commutator is $\star$-antisymmetric, is a $\star$-derivation in $\hat{\mathsf{A}}_\star$ and satisfies the $\star$-Jacobi identity

$$[F, G]_\star = -[\bar{R}^\alpha(G), \bar{R}_\alpha(F)]_\star \tag{4.10}$$

$$[F, G\star H]_\star = [F, G]_\star\star H + \bar{R}^\alpha(G)\star[\bar{R}_\alpha(F), H]_\star \tag{4.11}$$

$$[F, [G, H]_\star]_\star = [[F, G]_\star, H]_\star + [\bar{R}^\alpha(G), [\bar{R}_\alpha(F), H]_\star]_\star \tag{4.12}$$

Finally, recalling the definition of the $\mathcal{R}$-matrix it can be easily verified that

$$[F, G]_\star = F\star G - \bar{R}^\alpha(G)\star\bar{R}_\alpha(F) \tag{4.13}$$

which is indeed the $\star$-commutator in $\hat{\mathsf{A}}_\star$.

Evaluating the twisted commutator among the fields we find

$$\begin{aligned}
[\phi_\pm(x), \phi_\pm(y)]_\star &= [\phi_\pm(x), \phi_\pm(y)] \\
&\quad + \frac{i}{2}\theta^{-+}([\partial_-\phi_\pm(x), \partial_+\phi_\pm(y)] - [\partial_+\phi_\pm(x), \partial_-\phi_\pm(y)]) + O(\theta^2) \\
&= [\phi_\pm(x), \phi_\pm(y)] \qquad (4.14)\\
[\phi_\pm(x), \phi_\mp(y)]_\star &= 0 \qquad (4.15)
\end{aligned}$$

because $\phi_\pm$ are respectively functions of $x^\pm$ only. Thus, the twisted commutators are equal to the undeformed ones. We can compute the twisted commutators of the currents as well and we find that those too are undeformed, essentially for the same reason

$$\begin{aligned}
[J^\pm(x), J^\pm(y)]_\star &= [J^\pm(x), J^\pm(y)] \\
&\quad + \frac{i}{2}\theta^{-+}([\hat{-}J^\pm(x), \hat\partial_+ J^\pm(y)]\partial_-[\hat\partial_+ J^\pm(x), \hat\partial_- J^\pm(y)]) + O(\theta^2) \\
&= [J^\pm(x), J^\pm(y)] \qquad (4.16)\\
[J^\pm(x), J^\mp(y)]_\star &= 0 \qquad (4.17)
\end{aligned}$$

This doesn't mean however that $\star$-commutators of more complicated functionals are undeformed as well. Since the (usual) Leibniz rule doesn't hold anymore, when we put together functions their commutator will be different.

To verify, or disprove, quantum conformal invariance, we have to calculate the twisted commutators for the energy-momentum tensor components

$$\Theta^{\pm\pm}(x) = \gamma : J^\pm(x) \star J^\pm(x) : , \qquad (4.18)$$

where the normal ordering is defined in (2.11). In order to evaluate these commutators we need the commutation rules for the creation and annihilation operators. For fixed values of k, $a_\sigma(k)$ may be regarded as functionals of the coordinate fields

$$\begin{aligned}
a_-(k) &= 2|k^1| \int_{-\infty}^{\infty} dx^+ e^{ik^1x^+} \phi_+(x^+) \\
a_+(k) &= 2|k^1| \int_{-\infty}^{\infty} dx^- e^{ik^1x^-} \phi_-(x^-) \qquad (4.19)
\end{aligned}$$

By means of (4.6) we find

$$\hat\partial_\mp a_\pm = -ik^1 a_\pm \qquad \hat\partial_\pm a_\pm = 0 \qquad (4.20)$$

Therefore we can compute the $\star$-commutator of creation and annihilation operators using (4.9) and the appropriate expression of the twist (4.7). We find

$$\begin{aligned}
[a_\pm(k), a_\pm(k')]_\star &= [a_\pm(k), a_\pm(k')] + \frac{i}{2}\theta^{-+}([\hat\partial_- a_\pm(k), \hat\partial_+ a_\pm(k')] \\
&- [\hat\partial_+ a_\pm(k), \hat\partial_- a_\pm(k')]) + O(\theta^2) = [a_\pm(k), a_\pm(k')] \qquad (4.21)
\end{aligned}$$

because of (4.20), while

$$
\begin{aligned}
[a_\pm(k), a_\mp(k')]_\star &= [a_\pm(k), a_\mp(k')] + \frac{i}{2}\theta^{-+}([\hat\partial_- a_\pm(k), \hat\partial_+ a_\mp(k')] \\
&- [\hat\partial_+ a_\pm(k), \hat\partial_- a_\pm(k')]) + O(\theta^2) = e^{-\frac{i}{2}\theta^{-+}k^1 k'^1}[a_\pm(k), a_\mp(k')] = 0
\end{aligned} \tag{4.22}
$$

because $[a_\pm(k), a_\mp(k')] = 0$. Therefore the algebra of creation and annihilation operators is undeformed. Let us consider now the commutators of the stress-energy tensor components. We have

$$
\begin{aligned}
[\Theta^{++}(x), \Theta^{++}(y)]_\star &= [\Theta^{++}(x), \Theta^{++}(y)] + \frac{i}{2}\theta^{-+}\Big([\hat\partial_-\Theta^{++}(x), \hat\partial_+\Theta^{++}(y)] \\
&- [\hat\partial_+\Theta^{++}(x), \hat\partial_-\Theta^{++}(y)]\Big) + O(\theta^2) \\
&= [\Theta^{++}(x), \Theta^{++}(y)]
\end{aligned} \tag{4.23}
$$

only the zeroth order in θ survives, that is the undeformed commutator, because $\hat\partial_\pm\Theta^\pm(y)$ is zero. The same happens for the commutator of the Θ^{--} component

$$
\begin{aligned}
[\Theta^{--}(x), \Theta^{--}(y)]_\star &= [\Theta^{--}(x), \Theta^{--}(y)] + \frac{i}{2}\theta^{-+}\Big([\hat\partial_-\Theta^{--}(x), \hat\partial_+\Theta^{--}(y)] \\
&- [\hat\partial_+\Theta^{--}(x), \hat\partial_-\Theta^{--}(y)]\Big) + O(\theta^2) \\
&= [\Theta^{--}(x), \Theta^{--}(y)].
\end{aligned} \tag{4.24}
$$

Both the results had to be expected because the twist operator mixes left and right sectors; therefore surprises could come only from the mixed commutator

$$
[\Theta^{++}(x), \Theta^{--}(y)]_\star = [\Theta^{++}(x), \Theta^{--}(y)] + \frac{i}{2}\theta^{-+}[\hat\partial_-\Theta^{++}(x), \hat\partial_+\Theta^{--}(y)] + O(\theta^2). \tag{4.25}
$$

Let us compute the first order in the deformation parameter. We find

$$
\frac{i}{2}\theta^{-+}[\hat\partial_-\Theta^{++}(x), \hat\partial_+\Theta^{--}(y)] = \frac{i}{2}\theta^{-+}\frac{\partial}{\partial x^-}\frac{\partial}{\partial y^+}[\Theta^{++}(x), \Theta^{--}(y)] = 0 \tag{4.26}
$$

because $[\Theta^{++}(x), \Theta^{--}(y)] = 0$. The same holds true for all higher orders in θ, that is the mixed commutator is undeformed as well and we conclude that, thanks to the twist, the quantum conformal algebra is preserved.

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